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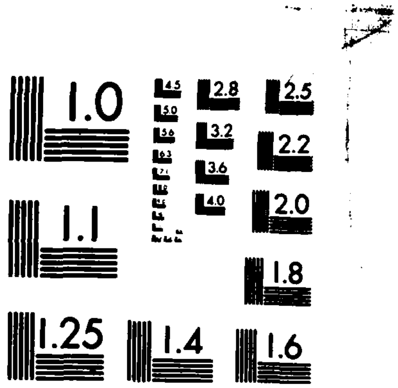

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EXTREMAL AND GAME-THEORETIC CHARACTERIZATIONS OF THE PROBABILISTIC APPROACH TO INCOME REDISTRIBUTION

by

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Abstract

In this paper we cast the problem of income redistribution in two different ways, one as a non-linear goal programming model, and the other as a game theoretic model. These two approaches give characterizations for the probabilistic approach suggested by Intriligator for this problem. All three approaches reinforce the linear income redistribution plan as a desirable mechanism of income redistribution.

Key Words: probabilistic approach, linear income system, non-linear goal programming, Kullback-Leibler information function, core of a game, Shapley value.

0. Introduction

This paper addresses the problem of optimal distribution of income utilizing three different approaches. The purpose is to unify these approaches and to show that in some important cases they are equivalent in the sense that we obtain the same optimal distribution regardless of which method is used.

The income distribution problem has many aspects which cannot be addressed in this paper. One aspect is to see how income and wealth are distributed and to explain the difference in their distribution [2]. Another aspect is the impact on society and economic efficiency of government measures to redistribute income and wealth [15,18]. There are also philosophical questions concerning income distribution e.g. what meaning should be attached to inequality of income? Is it desirable or not? What measures should be used to measure equality or inequality [2]?

Our concern in this paper is optimal distribution of income. The approaches taken in the past involve either the implications of certain axiomatizations of the concept of "equity", or, deductions from general or specific social welfare functions. In [1] Atkinson deduced, assuming a certain welfare function, that a linear income system as defined in (2.12) is optimal for that welfare function. Hammond in [8] showed that an "equality" income system as defined in (2.11) is optimal under assumptions which involve the level of utility functions and the type of welfare function used. The above represent typical efforts in the current trend for addressing optimal distribution of income. A different approach is suggested by Intriligator in [9]. He suggested the probabilistic approach which we shall review.

In Section 1 we will define the problem and the assumptions underlying the models to be presented. In Section 2 we will present the approach given in [9]. In Section 3 we will cast the problem as a goal programming model. Section 4 will contain a game theoretic approach. In Section 5 we will compare the three models and conclude this paper.

1.0 Statement of the Problem

The problem of income distribution may be stated as follows. Consider a society with m individuals the i^{th} of which has an initial income y_i^o. The distribution of income is summarized by the vector

(1.1) $$\underset{\sim}{y}^o = (y_1^o, y_2^o, \ldots, y_m^o).$$

The total level of income at this initial distribution is

(1.2) $$Y^o = \sum_{i=1}^{m} y_i^o$$

Let us define a final distribution, i.e. a distribution of income after adjustment according to a policy mechanism, by the following vector

(1.3) $$\underset{\sim}{y} = (y_1, y_2, \ldots, y_m)$$

The total level of income for the final distribution is

(1.4) $$Y = \sum_{i=1}^{m} y_i$$

The problem of income distribution is that of shifting the inital distribution in (1.1), assumed exogenously given, to the final distribution, $\underset{\sim}{y}$, in some desirable way.

It is more realistic to consider the problem as income redistribution rather than income distribution as suggested by Intriligator in [9] because in most societies, whether market-oriented or socialist, there is no mechanism for creating a "first-shot" income distribution. Rather there are usually mechanisms for influencing and modifying an existing income distribution through taxes, social security, public assistance, rationing, or other programs.

Thus the problem is one of starting from some pre-existing distribution of income and, by various combinations of mechanisms, creating a new income distribution, i.e. a redistribution.

Two assumptions will be made in order to simplify matters. The first is that any distribution does not affect the total level of income. Thus

$$Y^o = Y. \tag{1.5}$$

Realistically, redistribution can and does have an effect on total income through incentives, i.e. if the redistribution policy doesn't give incentives for earning then this tends to affect the total level of income in succeeding periods. The second assumption is that there are enough mechanisms available to achieve any final distribution $\underset{\sim}{y}$ from an initial distribution $\underset{\sim}{y}^o$, subject only to constraints (1.4) and (1.5) which may be written as

$$\sum_{i=1}^{m} y_i^o = \sum_{i=1}^{m} y_i, \; y_i \geq 0 \tag{1.6}$$

The income redistribution problem can be restated as, given the inital incomes y_i^o, choose for each individual i an increase or a decrease x_i so that final income is the original income plus this change in income

$$y_i = y_i^o + x_i \tag{1.7}$$

Then the redistribution is summarized by the vector

$$x = (x_1, x_2, \ldots, x_m), \text{ where } x_i \geq - y_i^o, \sum_{i=1}^{m} x_i = 0 \tag{1.8}$$

If a particular x_i is postive, it can be interpreted as a subsidy, and if it is negative it can be interpreted as a tax. Then the redistribution problem is to choose x_i in order to satisfy some conditions of optimality, as discussed in Section 2.

A second way to state the income redistribution problem is as a transfer problem from some individuals in the society to others in a way to achieve the final income distribution defined in (1.3). If we define x_{ij} as the amount transferred from individual i to j, then we will have

$$\sum_j x_{ij} = y_i^o, \qquad \forall_i \tag{1.9}$$

$$\sum_i x_{ij} = y_j, \qquad \forall_j \tag{1.10}$$

Then the redistribution problem is to choose x_{ij} in order to achieve $\underset{\sim}{y}$ in a desirable way, as discussed in Section 3.

A third way to state the income redistribution problem is to consider the m individuals in the society as involved in an m-person game. We would like then to formulate the game in such a way that some solution of this game produces the final income redistribution $\underset{\sim}{y}$ defined in (1.3) in a desirable way, as discussed in Section 4.

2.0 A Probabilistic Approach to Income Redistribution

The probabilistic approach to income distribution, as presented in [9], is based on the probabilistic approach to social choice in [7,11]. The connection with social choice is clear since income distribution is one of the most important issues in social choice.

In the probabilistic approach to social choice, it is assumed that each individual i has a probability vector q_i expressing his preference among n alternatives

$$q_i = (q_{i1}, q_{i2}, \ldots, q_{in}) \qquad i = 1,\ldots,m \tag{2.1}$$

where q_{ij} is the probability that individual i will choose alternative A_j if he could act alone in deciding among the alternatives. We assume three axioms:

i) existence of *social* probabilities,

ii) unanimity preservation for a loser (i.e. if all individuals choose a particular alternative with probability zero then so does society),

iii) strict and equal sensitivity of social probabilities to individual probabilities.

It follows that there is a unique rule to determine social probabilities P_j, where P_j is the probability that society will choose alternative A_j. This unique rule is the "average" rule which states that the social probabilities are simple averages of individual probabilities.

$$P_j = \frac{1}{m}\sum_{i=1}^{m} q_{ij} \tag{2.2}$$

This concept was applied to income redistribution by specifying alternatives A_j for final levels of income. There are many such alternatives. Indeed the set of alternatives is infinite. However, in [9] only discrete finite sets of alternatives were considered.

Consider the following extreme situation where we have m alternatives and alternative j is defined as

$$A_j: \; y_j = Y \text{ and } y_i = 0, \text{ for all } i \neq j \tag{2.3}$$

$$\text{i.e. } x_j = Y - y_j^o \text{ and } x_i = - y_i^o, \text{ for all } i \neq j$$

This is an extreme alternative for income redistribution since A_j refers to the situation in which all individuals other than j turn over their income to j. If individual j can obtain alternative A_j he will assign probability one to this alternative since it gives him all the income, i.e.

$$q_{ij} = \delta_{ij} = 1 \text{ or } 0 \text{ according as } i = j \text{ or not} \tag{2.4}$$

where δ_{ij} is the Kronecker delta. Under the average rule the society chooses A_j with probability P_j given by

$$P_j = \frac{1}{m} \tag{2.5}$$

implying that individual i has probability $1/m$ of getting all the income in the society. This societal choice is clearly one of equity in the sense that each individual has an equal chance of obtaining all the income in the society. Also each individual has a chance of $1 - \frac{1}{m}$ of receiving zero income under the alternatives considered here. This result is an extremely limited one due to the restrictions imposed on the alternatives.

Next suppose that individual i receives with probability one a base level income $\overline{y}_i$ considered as the subsistence level of income.[1] It is assumed the total level of income exceeds the sum of the base levels of income.

$$(2.6) \qquad Y = \sum_{i=1}^{m} y_i^o > \sum_{i=1}^{m} \overline{y}_i, \quad \text{or} \quad Y^s = Y - \overline{Y} > 0$$

where Y^s is surplus income and $\overline{Y}$ is the total of the base levels of income. Consider the case where the number of alternatives equals the number of individuals in the society and the alternatives are given by

$$(2.7) \qquad A_j : y_j = \overline{y}_j + Y^s \quad \text{and} \quad y_i = \overline{y}_i, \quad \text{for all} \quad i \neq j$$

$$\text{i.e.} \quad x_j = Y^s - (y_j^o - \overline{y}_j) \quad \text{and} \quad x_i = -(y_i^o - \overline{y}_i), \quad \text{for all} \quad i \neq j$$

This case corresponds to the situation in which each individual receives his subsistence level of income, but the j^{th} individual receives in addition all the surplus income. The individual probability vector is again given by

$$(2.8) \quad q_{ij} = \delta_{ij} = 1 \text{ or } 0 \text{ according as } i \neq j \text{ or not}$$

[1]The base level of income can be identified as the total expenditures on all base levels of goods and services in a linear expenditure system. See [10,14] for discussions of this system. Income tax laws generally identify base level of income as expenditures required for personal maintenance of the individual.

The average rule implies that the individual j after redistribution has the following income

$$
(2.9)\qquad \begin{cases} \bar{y}_j + Y^s & \text{with probability } \frac{1}{m} \\ \bar{y}_j & \text{with probability } 1 - \frac{1}{m} \end{cases}
$$

implying that all individuals receive their base level of income with probability 1, with individual j receiving all the surplus income with probability $1/m$. Now assume individuals are risk averse but will settle for expected values, given as

$$
(2.10)\qquad E(y_i) = \bar{Y}_j + \frac{1}{m} Y^s
$$

The resulting income redistribution is characterized by the equal of income received over the base levels of income. This redistribut[illegible]s referred to as the "equality income system" (EIS). In the equality system each individual receives his base level of income plus a proportionate share of the surplus income.

$$
(2.11)\qquad y^E_j = \bar{y}_j + \frac{1}{m} Y^s = \bar{y}_j + \frac{1}{m} (Y^o - \bar{Y}) = \bar{y}_j +
$$

$$
+ \quad \frac{1}{m} \Big(\sum_{i=1}^{m} y_i^o - \sum_{i=1}^{m} \bar{y}_i \Big) \quad \text{for all } j = 1, 2, \ldots, m.
$$

A generalization of the equality income system is the linear income system (LIS) defined by

$$
(2.12)\qquad y_j = \bar{y}_j + \beta_j Y^s \quad \text{for } j = 1, 2, \ldots, m.
$$

In this system each individual receives his base level of income plus a share of all income in excess of total base levels of income. This system is defined by the set of base levels $\bar{y}_i$ and shares. The shares, called "marginal income shares", satisfy the following conditions

$$\beta_j \geq 0, \quad \sum_{j=1}^{m} \beta_j = 1 \tag{2.13}$$

The equality income system is that special case of the linear income system with $\beta_j = 1/m$ for all $j = 1, 2, \ldots, m$. Another interesting redistribution system is the proportional linear income system (PLIS). It arises as the special case of the linear income system where

$$\beta_j = y_j^o/Y^o \quad \text{for all} \quad j = 1, 2, \ldots, m. \tag{2.14}$$

The redistribution systems obtained in this section by probabilistic constructs will be obtained by other approaches in Section 3 and 4.

3.0 Non-Linear Goal Programming Model

We would like tc emphasize that the model to be presented here is not dependent on assumptions of existence of utility functions or of social welfare functions. Atkinson in [1] and Champernowne in [3] employed such assumptions, however, and showed that we can obtain the linear income system given in (2.12) by solving the following problem:

$$\text{Minimize} \quad W = -\sum_{i=1}^{m} \beta_i \ln(y_i - \bar{y}_i) \tag{3.1}$$

subject to

$$\sum y_i = \sum y_i^o \tag{3.2}$$

$$y_i \geq 0$$

W is the Bergson welfare function, generated from the logarithmic utility function. If we apply the Kuhn-Tucker conditions to (3.1) we get

$$-\beta_i (y_i - \bar{y}_i)^{-1} = \lambda + \mu_i \tag{3.3}$$

$$y_i \mu_i = 0 \tag{3.4}$$

If $y_i > 0$ then $\mu_i = 0$ and $\beta_i/\lambda = \bar{y}_i - y_i$

$$\text{or, } y_i = \bar{y}_i - \beta_i/\lambda \ ,$$

$$\text{so that } \sum_i^m y_i = \sum_i^m \bar{y}_i - \sum_{i=1}^m \beta_i/\lambda$$

But $\sum \beta_i = 1$. Thus $Y^s = -\frac{1}{\lambda}$.

Thus we obtain $y_i = \bar{y}_i + \beta_i Y^s$, the linear income system.

The difficulties with the Atkinson-Champernowne approach are the underlying assumptions of existence of utility functions and social welfare functions.

There are many other measures of income inequality in the economic literature. One of these, the Kullback-Leibler information theoretic measure, has many desirable properties.[2] It is derived from the expected relative information concept, and it will be used as our objective to minimize. Using it we reformulate the income redistribution problem as follows. Let

x_{ij} = amount of income transferred from individual i to individual j in redistribution . Then we have the following:

$$\sum_{j=1}^m x_{ij} = y_i^o \qquad i = 1, 2, \ldots, m. \tag{3.5}$$

$$\sum_{i=1}^m x_{ij} = y_j \qquad i = 1, 2, \ldots, m. \tag{3.6}$$

[2]For more on the properties of the Kullback-Leibler information function as a measure of income inequality, see Chapter III and IV of [16] and [6,17].

Now we define income redistribution as the solution to the following extremal problem:

(3.7) $$\text{Minimize} \sum_i \sum_j x_{ij} \ln (x_{ij})$$

subject to equation (3.5) and (3.6) and all $x_{ij} \geq 0$.

Applying Lagrange multipliers, we obtain as the first-order conditions

(3.8) $$\ln(x_{ij}) + 1 + \lambda_i = 0$$

$$\sum_{j=1}^{n} x_{ij} = y_i^o$$

Solving for x_{ij}, we obtain

$$x_{ij} = e^{-\lambda_i - 1}$$

Using equation (3.5) $\sum_j x_{ij} = y_i^0 = me^{-\lambda_i - 1}$

Hence $$\ln y_i^o/m = -\lambda_i - 1$$

and $$x_{ij}^* = y_i^o/m$$

The optimal redistribution of income is then

$$\sum_i x_{ij}^* = \sum_i y_i^o/m = \frac{1}{m} Y^o = \frac{1}{m} Y ,$$

a uniform distribution of total income. To be more realistic we should add constraints such as that each individual has to attain his subsistence level of income. Also we should modify our goals in the objective function. In this way we suggest the following extremal problem to characterize income redistribution:

(3.9) $$\text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{m} x_{jn} \ln(x_{ij}/c_{ij})$$

subject to (3.5) and

(3.10)
$$\sum_i x_{ij} \geq \bar{y}_j$$
$$x_{ij} \geq 0$$

Applying the Kuhn-Tucker conditions yields

(3.11)
$$\ell n\ (x_{ij}/c_{ij}) + 1 = \lambda_i + \mu_j$$
$$\sum_j x_{ij} = y_i^o$$

(3.12)
$$(-\sum_i x_{ij} + y_j)\mu_j = 0$$
$$\mu_j \geq 0$$

If we assume $\sum_i^m x_{ij} > \bar{y}_j$, then we obtain the complementary slackness condition $\mu_j = 0$ for all j. Then

$$\ell n\ (x_{ij}/c_{ij}) = -1 + \lambda_i$$

$$\text{and} \qquad x_{ij} = c_{ij}\ e^{-1+\lambda_i}$$

Summing over j

$$\sum_j x_{ij} = e^{-1+\lambda_i}\left(\sum_{j=1}^m c_{ij}\right)$$

Let

$\sum_{j=1}^m c_{ij} \triangleq c_{i\cdot}$, and $\sum_{i=1}^m c_{ij} \triangleq c_{\cdot j}$. Then we have

$$\lambda_i - 1 = \ell n\ (y_i^o/c_{i\cdot})$$
$$x_{ij}^* = c_{ij}(y_i^o/c_{i.}).$$

Optimal income redistribution is then given by

$$y_j = \sum_i x^*_{ij} = \sum_i \left(\frac{c_{ij}}{c_{j.}}\right) y^o_i$$

The problem of income redistribution in this setting is reduced to the choice of the c_{ij} matrix. For each matrix $C \triangleq (c_{ij})$ corresponds an income redistribution pattern. The C matrix may be taken to represent a policy matrix.

The following examples show some income redistribution patterns derived from the extremal problem with different choice of C. Suppose $C = I$, the identity matrix

i.e.
$$c_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

This matrix results in the status quo income distribution system presented in [9] since we have

$$\begin{aligned} x^*_{ij} &= \frac{c_{ij}}{c_{i.}} y^o_i &&= 0\ y^o_i, \quad \text{if} \quad i \neq j \\ &= \frac{1}{1} y^o_i &&= y^o_i, \quad \text{if} \quad i = j \end{aligned}$$

so that $y_j = y^o_j$.

Each individual transfers all of his income to himself.

Next let $C = E$, the matrix of ones,

i.e.
$$c_{ij} = 1, \ \forall i,j$$

This matrix results in the uniform redistribution pattern

$$x^*_{ij} = \frac{1}{m} y^o_i$$

$$y_i = \sum_{i=1}^{m} x^*_{ij} = \frac{1}{m} Y^o = \frac{1}{m} Y$$

An interesting example which gives rise to the linear income system is the following:

Let $C = (c_{ij})$ where

$$c_{ij} = \begin{cases} k_j \dfrac{(y_i^o - \overline{y}_i)}{y_i^o} c_{i\cdot} & \text{for } i \neq j \\ \dfrac{[k_j\ (y_j^o - \overline{y}_j) + \overline{y}_j]c_{j\cdot}}{y_j^o} & \text{for } i = j \end{cases}$$

where k_j is any non-negative number. If we set $c_{i\cdot} = 1$, we get for $C = (c_{ij})$

$$c_{ij} = \begin{cases} \dfrac{k_j\ (y_i^o - \overline{y}_i)}{y_i^o}, & i \neq j \\ , & \\ \dfrac{k_j\ (y_j^o - \overline{y}_j) + \overline{y}_j}{y_j^o}, & i = j \end{cases}$$

The final income distribution will be

$$y_j = \sum_{i=1}^{m} \frac{c_{ij}}{c_{i.}} y_i^o = \sum_{\substack{i=1 \\ i\neq j}}^{m} k_j (y_i^o - \overline{y}_i) + k_j (y_j^o - \overline{y}_j) + \overline{y}_j$$

$$= \overline{y}_j + k_j \sum_{\substack{i=1 \\ i\neq j}}^{m} (y_i^o - \overline{y}_i) = \overline{y}_j + k_j Y^s, \quad j = 1, 2, \ldots, m$$

i.e. the linear income system given in (2.12). We note that we must have $\sum_{j=1}^{m} k_j = 1$. For that consider

$$1 = c_{i.} = \sum_{j=1}^{m} c_{ij} = \frac{y_i^o - \overline{y}_i}{y_i^o} \sum_{j=1}^{m} k_j + \frac{\overline{y}_i}{y_i^o}$$

Thereby

$$\frac{y_i^o - \bar{y}_i}{y_i^o} = \frac{y_i^o - \bar{y}_i}{y_i^o} \sum_{j=1}^{m} k_j \Rightarrow \sum^{m} k_j = 1 \quad \text{as noted.}$$

Thus also the k_j's correspond to the β_j's in (2.12).

The extremal principle formulation has another advantage over the probabilistic approach in that we can easily add whatever other realistic inequality conditions we might want to satisfy. Another important advantage of the extremal formulation in (3.9) is that that problem and its duality states are well characterized by A. Charnes, W. W. Cooper and L. Seiford in [4].

4.0 <u>A Game Theoretic Approach to Income Redistribution</u>

The results of section 2 and 3 can be obtained by still another approach. We will formulate the income redistribution problem in terms of a game with m players. The problem lends itself naturally to definition as a cooperative game. Then, various concepts of solution in game theory can be applied to give optimal income redistributions.

The game is assumed given in characteristic function form. A characteristic function of a game is a mapping from the set of all subsets of the set of all players $M \equiv \{1, 2, \ldots, m\}$ into the real numbers satisfying the following:

(4.1) $V(\emptyset) = 0$

(4.2) $V(S \cup T) \geq V(S) + V(T)$, whenever $S \cap T = \emptyset$ where $S, T \subseteq M$.

This latter property is called "superadditivity". For more about games in characteristic function form see [12,13,19].

There are many concepts of solution for a game. The most popular concepts are the von Neumann-Morgenstern solution, the core, the Shapley value, and the nucleolus. In this section we shall apply the Shapley value solution. For more about concepts of solution see [5,12,13,19]. The Shapley value for an m-person game with characteristic function V is defined as

$$\phi_i = \sum_{\substack{T \subseteq M \\ i \in T}} \frac{(t-1)!(n-t)!}{n!} [V(T) - V(T - \{i\})] \tag{4.3}$$

where $t = |T|$, the number of elements in T. ϕ_i is to be the final income y_i for individual i from the m-person game with characteristic function V. For more about the Shapley value see [12,13].

A cooperative game with an additive characteristic function is called an "inessential" game. Other cooperative games are called "essential".

An imputation $x = (x_1, \ldots, x_m)$ for an m-person game with characteristic function V is a vector satisfying

$$x_i \geq V(i) \quad \text{and} \quad \sum x_i = V(M) \tag{4.4}$$

A vector $x = (x_1, \ldots, x_m)$ is in the core of a game iff it satisfies the following

$$x_i \geq V(i), \quad \sum_{i \in T} x_i \geq V(T), \quad \sum_{i=1}^{m} x_i = V(M), \tag{4.4}$$

for every $T \subseteq M$.

<u>Theorem 4.1</u>

An inessential game has only one imputation given by $(V(1), \ldots, V(m))$ and this imputation is the core and the Shapley value.

<u>Proof</u>

Given any imputation $x = (x_1, x_2, \ldots, x_n)$ can be expressed as $x_i = V(i) + a_i$, $a_i \geq 0$, $i = 1, 2, \ldots, m$. Then we have

$\sum_{i-1}^{m} x_i = \sum_{i=1}^{m} V(i) + \sum_{i=1}^{m} a_i$. But since the game is inessential we must have

$$\sum_{i=1}^{m} V(i) = V(M).$$ Also from the definition of imputation we have

$$\sum_{i=1}^{m} x_i = V(M).$$ This implies $\sum a_i = 0$, and since all $a_i \geq 0$ then $a_i = 0$ for all i. Hence $\underline{x} = (x_1, \ldots, x_m) = (V(1), \ldots, V(m))$.

We notice next that the unique imputation $(V(1), \ldots, V(m))$ satisfies (4.5). Hence it is in the core and the core consists of this one imputation.

Since this is the only imputation, and the Shapley value exists for every m-person cooperative game, it must be the Shapley value.

Q.E.D.

Theorem 4.2

The linear income system defined in (2.12) is the imputation given by the core and the Shapley value of an inessential game.

Proof

Consider the m-individuals in the society as players in the game with characteristic function

$$V(T) = \sum_{i\varepsilon T} \overline{y}_i + Y^s \sum_{i\varepsilon T} \beta_i, \qquad T \subseteq M \tag{4.6}$$

where $M = \{1,2,...,m\}$ is the set of players and $\overline{y}_i$, Y^s, and β are as defined in Section 2. Note that $V(M) = Y^o$.

The game defined by (4.6) is inessential, since for all $R, T \subseteq M$ with $R \cap T = \phi$, we have

$$V(R \cup T) = V(R) + V(T) = \sum_{i\varepsilon R} V(i) + \sum_{j\varepsilon T} V(j) \tag{4.7}$$

By theorem (4.1) the Shapley value for the game defined in (4.6) is the same as the core and is given by

$$\phi_i = V(i) = \overline{y}_i + \beta_i Y^s \tag{4.8}$$

Hence the Shapley value and the core of this game constitute the linear income system defined in (2.12).

Q.E.D.

As noted an inessential game has precisely one imputation. An essential game has in general an infinite number of imputations. To formulate the income redistribution problem in terms of essential games will help in identifying different possible income redistribution patterns.

Thus we consider an m-person game with characteristic function

$$V(T) = \sum_{i \in T} \bar{y}_i + C_T Y^s, \qquad T \subseteq M \tag{4.9}$$

where M is the set of all m-players and C_T is a constant which reflects the power and the contribution of subset T of the players. More generally C_T may be considered as a function of each individual's "power" in coalition T. The C_T should satisfy the following conditions.

$$C_R \geq 0, \quad C_R \leq C_T \quad \text{iff} \quad R \subseteq T, \quad C_M = 1. \tag{4.10}$$

(4.11) C_T is invariant under permutation of elements of T within T.

Permuting elements with the same initial income, one could write

$$c_{ij} = c_{ji} \quad \text{or} \quad c_{ijk} = c_{jik} = c_{kji} \quad \text{and so on}$$

where the subscripts refer to players in T.

We note that the characteristic function in (4.6) is a special case of (4.9). By taking $C_T = \sum_{i \in T} \beta_i$ we get (4.6).

One type of (4.9) characteristic function is obtained by taking

$$C_T = \prod_{i \in T} y_i^o \Big/ \prod_{i \in M} y_i^o = \frac{1}{\prod_{i \notin T} y_i^o} \tag{4.12}$$

To satisfy (4.10), y_i^o should be greater than 1 for each i. We can scale the initial incomes to have $y_i^o \geq 1$ for all $i \in M$.

Example:

Consider a society with three members. Suppose the initial distribution is (5,10,20), i.e. $y_1^o = 5$, $y_2^o = 10$, and $y_3^o = 20$. To redistribute the income, we define a characteristic function for the game and then compute its Shapley value. Let $\overline{y} = 7$, for all $i = 1, 2, 3$. Then $Y^s = \sum_{i=1}^{3} y_i^o - \sum_{i=1}^{3} \overline{y}_i = 35 - 21 = 14$. Using the definitions of (4.9) and (4.12), we obtain the characteristic function

(4.13) $$V(1) = 7.07, \quad V(2) = 7.14, \quad V(3) = 7.28,$$

$$V(1,2) = 14.7, \quad V(1,3) = 15.4, \quad V(2,3) = 15.8$$

$$V(1,2,3) = 35$$

The redistribution plan given by the Shapley value for this game is

$$y_1 = 11.56, \qquad y_2 = 11.63, \qquad y_3 = 11.81$$

We compare this next to other income redistribution systems given in Section 2.

The equality income system yields

$$y_i = 11.66 \quad \text{for all} \quad i = 1, 2, 3$$

while the proportional linear income system yields

$$y_1 = 9, \qquad y_2 = 11, \qquad y_3 = 15$$

The redistribution plan given by the Shapley value for the game in (4.13) is in the core of the game, but it is biased toward low income individuals. The linear income system does give a reasonable redistribution plan. This plan as we know from theorem (4.2) can also be generated from an inessential game.

5.0 Conclusion

We have examined three different approaches to the problem of optimal redistribution of income -- the probabilistic, extremal, and game theoretic approaches. The extremal and the game theoretic approaches each yield results similar to those of the probabilistic approach, including, in particular, as one important special case, the linear income system in (2.12).

Thus the linear income system, in which each individual receives his base level of income plus a share of all income in excess of total base levels of income can be interpreted as an outcome of a probabilistic social choice mechanism, or, alternatively, as an implication of the extremal problem using the Kullback-Leibler information function with certain weights, or alternatively, as the Shapley value or core of a game theoretic problem with a certain characteristic function. These alternative interpretations provide reinforcing justifications for consideration of the linear income system as a desirable mechanism of income redistribution.

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In this paper we cast the problem of income redistribution in two different ways, one as a non-linear goal programming model, and the other as a game theoretic model. These two approaches give characterizations for the probabilistic approach suggested by Intriligator for this problem. All three approaches reinforce the linear income redistribution plan as a desirable mechanism of income redistribution.

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